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The Frobenius map, rank 2 vector bundles and Kummer's quartic surface in characteristic 2 and 3

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Abstract

Let X be a smooth projective curve of genus $g \geqslant 2$ defined over an algebraically closed field k of characteristic $p > 0$. Let $\mathrm{M}_X(r)$ be the moduli space of semi-stable rank r vector bundles with fixed trivial determinant. The relative Frobenius map $F : X \to X_1$ induces by pull-back a rational map $V : \mathrm{M}_{X_1}(r) \to \mathrm{M}_X(r)$. We determine the equations of V in the following two cases (1) $(g, r, p) = (2, 2, 2)$ and X nonordinary with Hasse–Witt invariant equal to 1 (see math.AG/0005044 for the case X ordinary), and (2) $(g, r, p) = (2, 2, 3)$. We also show the existence of base points of V, i.e., semi-stable bundles E such that F^*E is not semi-stable, for any triple (g, r, p).

MSC: Primary 14H60; 14D20; Secondary 14H40

1. Introduction

Let X be a smooth projective curve of genus 2 defined over an algebraically closed field k of characteristic $p > 0$. The moduli space M_X of semi-stable rank 2 vector bundles with fixed trivial determinant is isomorphic to the linear system $|2\Theta| \cong \mathbb{P}^3$ over $\mathrm{Pic}^1(X)$ and the k-linear relative Frobenius map $F : X \to X_1$ induces by

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pull-back a rational map (the Verschiebung)

$$\begin{array}{ccc} \mathrm{M}_{X_1} & \xrightarrow{V} & \mathrm{M}_X \\ {\scriptstyle D}\downarrow & & \downarrow{\scriptstyle D} \\ |2\Theta_1| & \xrightarrow{\tilde{V}} & |2\Theta| \end{array} \tag{1.1}$$

Here X_1 denotes the curve $X \times_{k,\sigma} k$ where $\sigma :\ \mathrm{Spec}(k) \to \mathrm{Spec}(k)$ is the Frobenius of k (see e.g. [R, Section 4.1]). The vertical maps D of (1.1) are isomorphisms and the Verschiebung $V : E \mapsto F^*E$ coincides via D with a rational map $\tilde{V}$ given by polynomial equations of degree p (Proposition 7.2). The Kummer surfaces Kum_X and Kum_{X_1} are canonically contained in the linear systems $|2\Theta|$ and $|2\Theta_1|$ and coincide with the semi-stable boundary of the moduli spaces M_X and M_{X_1}. Moreover $\tilde{V}$ maps Kum_{X_1} onto Kum_X.

Our interest in diagram (1.1) comes from questions related to the action of the Frobenius map on vector bundles like, e.g. surjectivity of V, density of Frobenius-stable bundles, loci of Frobenius-destabilized bundles (see [LP]). For general (g, r, p) it is shown [MS] that for an ordinary curve X there exists a nonempty open subset of $\mathrm{M}_{X_1}(r)$ where V is étale, hence V is dominant. In the case $(r, p) = (2, 2)$ and g general, loci of Frobenius-destabilized bundles are studied in [JRXY]. In [LP] we made use of the exceptional isomorphism $D :\ \mathrm{M}_X \to |2\Theta|$ in the genus 2, rank 2 case and determined the equations of $\tilde{V}$ when X is an ordinary curve and $p = 2$, which allowed us to answer the above-mentioned questions. In this paper we obtain the equations of $\tilde{V}$ in two more cases:

(1) $p = 2$ and X nonordinary with Hasse–Witt invariant equal to 1,
(2) $p = 3$ and any X.

In case (1) we consider a family $\mathscr{X}$ of genus 2 curves parameterized by a discrete valuation ring with ordinary generic fiber $\mathscr{X}_\eta$ and special fiber isomorphic to X. We obtain the equations of $\tilde{V}$ for X (Theorem 5.1) by specializing the quadrics P_{ij} defining the Verschiebung $V_\eta :\ \mathrm{M}_{\mathscr{X}_{1\eta}} \to \mathrm{M}_{\mathscr{X}_\eta}$ associated to $\mathscr{X}_\eta$ (Section 5). In order to determine the limit of the P_{ij}'s, we use the explicit formulae (Proposition 3.1) of the coefficients of the P_{ij}'s, which coincide with the coefficients of Kummer's quartic surface $\mathrm{Kum}_{\mathscr{X}_\eta}$, in terms of the coefficients of an affine equation of the ordinary curve $\mathscr{X}_\eta$. As in the ordinary case we easily deduce from the equations of $\tilde{V}$ a full description of the Verschiebung V (Proposition 5.4).

In case (2) we show that the cubic equations of $\tilde{V}$ are given by the polar equations of a Kummer surface $S \subset |2\Theta_1|$ (Theorem 6.1). Moreover S is isomorphic to Kum_X and the 16 nodes of S correspond to the 16 base points of $\tilde{V}$. We deduce that V is surjective and of degree 11 (Corollary 6.6).

In the appendix we show that over *any* smooth curve X of genus $g \geqslant 2$ defined over an algebraically closed field of characteristic $p > 0$ and for any integer $r \geqslant 2$, there exist Frobenius-destabilized bundles of rank r, i.e., semi-stable bundles E such that

F^*E is not semi-stable (Theorem 7.4). This improves Gieseker's theorem [G], which asserts existence of Frobenius-destabilized bundles over a general curve.

2. Preliminaries on genus 2 curves in characteristic 2

We consider a smooth *ordinary* curve X of genus 2 defined over a field K of characteristic 2. We denote by $\bar{K}$ the algebraic closure of K, by $K(X)$ the function field of X, by JX the Jacobian of X and by ω_X the canonical line bundle of X. In this section we recall some results on equations and moduli of the curve X, which will be applied in Section 5 in the case $K = k((t))$.

2.1. Weierstrass points

After taking a finite extension of K and applying an automorphism of $\mathbb{P}^1_K$ we can assume that the three Weierstrass points of X are $0, 1$ and ∞. We consider the birational Abel–Jacobi map

$$\mathrm{AJ} : \mathrm{Sym}^2(X) \to JX, \quad P_1 + P_2 \mapsto \mathcal{O}_X(P_1 + P_2) \otimes \omega_X^{-1}. \tag{2.1}$$

We observe that the three nonzero elements $[0], [1], [\infty]$ of the group scheme of 2-torsion points $JX[2]$ are K-rational and are given by

$$[0] := \mathrm{AJ}(1 + \infty), \quad [1] := \mathrm{AJ}(0 + \infty), \quad [\infty] := \mathrm{AJ}(0 + 1). \tag{2.2}$$

For later use we mention that the sheaf of locally exact differential forms (see [R, Section 4.1]) equals

$$B = \mathcal{O}_X(0 + 1 + \infty) \otimes \omega_X^{-1}.$$

We recall that B is a theta-characteristic of X, i.e., $B^{\otimes 2} \cong \omega_X$.

2.2. Level 2 structures

A level 2 structure is an isomorphism $\psi : JX[2] \xrightarrow{\sim} \mathbb{F}_2^2$. Note that two level 2 structures differ by an automorphism of $\mathbb{F}_2^2$, i.e., an element of $\mathrm{GL}(2, \mathbb{F}_2) = \mathfrak{S}_3$, where $\mathfrak{S}_3$ is the symmetric group. It is well-known that a level 2 structure ψ is equivalent to an ordering of the Weierstrass points of X. We refer, e.g. to [DO, p. 141] for the characteristic zero case, which can easily be adapted to the characteristic two case. Because of the choices involved in the degeneration of X to a nonordinary curve (Section 4.3), we consider the ordering $1, \infty, 0$ of the Weierstrass points. With the notation of (2.2) the corresponding level 2 structure ψ is given by

$$\psi([0]) = (1, 0), \quad \psi([1]) = (0, 1), \quad \psi([\infty]) = (1, 1). \tag{2.3}$$

A level 2 structure ψ allows us to construct [LP, Section 2] the theta basis $\{X_g\}_{g \in \mathbb{F}_2^2}$ of the space $H^0(JX, 2\Theta)$. We denote the four sections by X_B, X_0, X_1, X_∞ and introduce the rational functions $Z_\bullet \in k(JX)$ defined by

$$Z_0 = \frac{X_0}{X_B}, \quad Z_1 = \frac{X_1}{X_B}, \quad Z_\infty = \frac{X_\infty}{X_B}. \tag{2.4}$$

We recall that X_B is the theta function whose associated nonreduced zero divisor equals $2\Theta_B$, with

$$\operatorname{supp}(\Theta_B) := \{L \in JX \mid h^0(X, L \otimes B) \geqslant 1\}.$$

2.3. Birational models

Let X be an ordinary smooth curve of genus 2 defined over $\bar{K}$ and ψ a level 2 structure. It follows from [L, p. 28] and [B, Proposition 1.4] that the pair (X, ψ) is *uniquely* represented by an affine equation of the form

$$y^2 + x(x+1)y = x(x+1)(ax^3 + (a+b)x^2 + cx + c), \tag{2.5}$$

with $a, b, c \in \bar{K}$. Moreover if X is defined over K, the coefficients a, b, c lie in a finite extension of K. The next lemma is an immediate consequence of [B, Proposition 1.5].

Lemma 2.1. *The curve X defined by Eq.* (2.5) *is smooth if and only if $abc \neq 0$.*

Let $\tilde{\mathcal{M}}_3$ denote the moduli space parameterizing pairs (X, ψ) of smooth ordinary genus 2 curves X defined over $\bar{K}$ equipped with a level 2 structure. It follows from the previous remark that $\tilde{\mathcal{M}}_3$ is an affine variety,

$$\tilde{\mathcal{M}}_3 = \operatorname{Spec} \bar{K}\left[a, b, c, \frac{1}{abc}\right].$$

Fixing the curve X, the symmetric group $\mathfrak{S}_3$ acts naturally on the level 2 structures ψ. It can be shown that this $\mathfrak{S}_3$-action on $\tilde{\mathcal{M}}_3$ coincides with the permutation action of $\mathfrak{S}_3$ on the coefficients a, b, c.

2.4. Normal form

We introduce the rational function $Y \in K(X)$ defined by $Y = \frac{y}{x(x+1)}$. Then (2.5) becomes

$$Y^2 + Y = R(x), \quad \text{with } R(x) = \frac{ax^3 + (a+b)x^2 + cx + c}{x(x+1)}. \tag{2.6}$$

We also observe that, given a polynomial $S \in \bar{K}[x]$, the involution $i_S : \mathbb{A}^2_{\bar{K}} \to \mathbb{A}^2_{\bar{K}}$, $(x, y) \mapsto (x, y + x(x+1)S)$ transforms Eq. (2.6) of X into $Y^2 + Y = R + S^2 + S$.

2.5. Kummer's quartic equation

For a pair (X, ψ) it has been shown in [LP, Proposition 4.1] that there exist constants $\lambda_0, \lambda_1, \lambda_\infty \in \bar{K}$ such that the following equality holds in $K(JX)$:

$$\lambda_0^2(Z_0^2 + Z_1^2 Z_\infty^2) + \lambda_1^2(Z_1^2 + Z_0^2 Z_\infty^2) + \lambda_\infty^2(Z_\infty^2 + Z_0^2 Z_1^2) + \lambda_0\lambda_1\lambda_\infty Z_0 Z_1 Z_\infty = 0. \quad (2.7)$$

The constants $\lambda_0, \lambda_1, \lambda_\infty$ are related via ψ (2.3) to the $\{\lambda_g\}_{g \in \mathbb{F}_2^2}$ used in [LP, Proposition 4.1] as follows: $\lambda_0 = \frac{\lambda_{10}}{\lambda_{00}}$, $\lambda_1 = \frac{\lambda_{01}}{\lambda_{00}}$, $\lambda_\infty = \frac{\lambda_{11}}{\lambda_{00}}$.

3. The coefficients $\lambda_\bullet$ of Kummer's quartic surface Kum_X for an ordinary curve X

In this section we determine the coefficients $\lambda_\bullet$ of the Kummer surface Kum_X for an ordinary curve X. This result will be used in the proof of Theorem 5.1 (Section 5).

Proposition 3.1. *Given a curve X with a level 2 structure ψ represented by an affine equation (2.5) with $a, b, c \in K$. Then the coefficients of Eq. (2.7) of its Kummer surface Kum_X are*

$$\lambda_0^2 = \frac{1}{ab}, \quad \lambda_1^2 = \frac{1}{ac}, \quad \lambda_\infty^2 = \frac{1}{bc}.$$

Let $\{x_g\}$ be the dual basis of the theta basis $\{X_g\}_{g \in \mathbb{F}_2^2}$ of $|2\Theta| = \mathbb{P}^3$. Then the homogeneous equation of Kum_X is

$$c(x_{00}^2 x_{10}^2 + x_{01}^2 x_{11}^2) + b(x_{00}^2 x_{01}^2 + x_{10}^2 x_{11}^2) + a(x_{00}^2 x_{11}^2 + x_{01}^2 x_{10}^2) + x_{00}x_{01}x_{10}x_{11} = 0.$$

The idea of the proof is to consider the pull-back of the rational functions $Z_\bullet$ (2.4) by the Abel–Jacobi map (2.1) to the symmetric product $\mathrm{Sym}^2(X)$ and to do the computations in the function field $K(S^2X) \hookrightarrow K(X \times X)$. Since X is given by Eq. (2.5), we have natural coordinates x_1, y_1 and x_2, y_2 on $X \times X$. For notational convenience we introduce $Y_i = \frac{y_i}{x_i(x_i+1)}$, for $i = 1, 2$.

We will use the following two lemmas.

Lemma 3.2. *Suppose that there exist polynomials $A, B \in K[x_1, x_2]$ which satisfy*

$$(Y_1 + Y_2)A(x_1, x_2) = B(x_1, x_2). \quad (3.1)$$

Then $A = B = 0$.

Proof. Squaring relation (3.1) and using (2.6) leads to the equation

$$(Y_1 + Y_2)A^2 + (R(x_1) + R(x_2))A^2 + B^2 = 0.$$

Applying again (3.1), the first term transforms into AB. Clearing denominators, we arrive at a polynomial equation, which only holds if $A = B = 0$ (e.g. by taking the total degree of A and B). □

Lemma 3.3. *The pull-back by the Abel–Jacobi map* AJ *of the rational function* $Z_\infty \in K(JX)$ *equals*

$$\mathrm{AJ}^*(Z_\infty) = \alpha_\infty \frac{P(x_1, x_2)}{(Y_1 + Y_2)^2}, \quad \text{with } P(x_1, x_2) = \frac{(x_1 + x_2)^2}{x_1 x_2 (x_1 + 1)(x_2 + 1)}.$$

Similarly we have

$$\mathrm{AJ}^*(Z_0) = \alpha_0 \frac{P(x_1, x_2)}{(Y_1 + Y_2)^2} x_1 x_2, \quad \mathrm{AJ}^*(Z_1) = \alpha_1 \frac{P(x_1, x_2)}{(Y_1 + Y_2)^2} (x_1 + 1)(x_2 + 1),$$

for some nonzero constants $\alpha_0, \alpha_1, \alpha_\infty \in K$.

Proof. The first equality follows immediately from Theorem 2 [AG] and the other two from Proposition 5 [AG]. □

Proof of Proposition 3.1. We write $Q = \frac{P(x_1, x_2)}{(Y_1 + Y_2)^2}$. Using Lemma 3.3, the pull-back to $\mathrm{Sym}^2(X)$ of Eq. (2.7) equals

$$\begin{aligned}
&\lambda_0^2[\alpha_0^2 x_1^2 x_2^2 Q^2 + \alpha_1^2 \alpha_\infty^2 (x_1 + 1)^2 (x_2 + 1)^2 Q^4] \\
&\quad + \lambda_1^2[\alpha_1^2 (x_1 + 1)^2 (x_2 + 1)^2 Q^2 + \alpha_0^2 \alpha_\infty^2 x_1^2 x_2^2 Q^4] \\
&\quad + \lambda_\infty^2[\alpha_\infty^2 Q^2 + \alpha_0^2 \alpha_1^2 x_1^2 x_2^2 (x_1 + 1)^2 (x_2 + 1)^2 Q^4] \\
&\quad + \lambda_0 \lambda_1 \lambda_\infty [\alpha_0 \alpha_1 \alpha_\infty x_1 x_2 (x_1 + 1)(x_2 + 1) Q^3] = 0.
\end{aligned}$$

Dividing by Q^2 and multiplying by $(Y_1 + Y_2)^4$, we obtain

$$\begin{aligned}
&(Y_1 + Y_2)^4 [\lambda_0^2 \alpha_0^2 x_1^2 x_2^2 + \lambda_1^2 \alpha_1^2 (x_1 + 1)^2 (x_2 + 1)^2 + \lambda_\infty^2 \alpha_\infty^2] \\
&\quad + (Y_1 + Y_2)^2 [\lambda_0 \lambda_1 \lambda_\infty \alpha_0 \alpha_1 \alpha_\infty P x_1 x_2 (x_1 + 1)(x_2 + 1)] + S = 0, \qquad (3.2)
\end{aligned}$$

where S is the sum of the remaining terms (not containing Y_1, Y_2). After taking the square root (note that the entire expression is a square of a polynomial in the x_i's and Y_i's), applying (2.6) and Lemma 3.2, we obtain that the coefficients of $(Y_1 + Y_2)^2$

and $(Y_1 + Y_2)^4$ are the same, i.e.,

$$\lambda_0^2\alpha_0^2x_1^2x_2^2 + \lambda_1^2\alpha_1^2(x_1+1)^2(x_2+1)^2 + \lambda_\infty^2\alpha_\infty^2 = \lambda_0\lambda_1\lambda_\infty\alpha_0\alpha_1\alpha_\infty P x_1x_2(x_1+1)(x_2+1).$$

An easy computation shows that this equality holds only if

$$\lambda_0\alpha_0 = \lambda_1\alpha_1 = \lambda_\infty\alpha_\infty = 1.$$

Now we replace the α's and the sum S by their expressions in the square root of Eq. (3.2)

$$(Y_1 + Y_2)^2(x_1+x_2) + (Y_1+Y_2)(x_1+x_2) + P\left[\frac{\lambda_0}{\lambda_1\lambda_\infty}(x_1+1)(x_2+1) + \frac{\lambda_1}{\lambda_0\lambda_\infty}x_1x_2 + \frac{\lambda_\infty}{\lambda_0\lambda_1}x_1x_2(x_1+1)(x_2+1)\right] = 0.$$

We introduce the constants $\mu_0, \mu_1, \mu_\infty \in K$ defined by $\mu_\infty = \frac{\lambda_\infty}{\lambda_0\lambda_1}, \mu_0 = \frac{\lambda_0}{\lambda_1\lambda_\infty}, \mu_1 = \frac{\lambda_1}{\lambda_0\lambda_\infty}$. The previous equality becomes after replacing P by its expression and dividing by (x_1+x_2)

$$\left[Y_1^2 + Y_1 + \frac{\mu_0}{x_1} + \frac{\mu_1}{x_1+1} + \mu_\infty x_1\right] + \left[Y_2^2 + Y_2 + \frac{\mu_0}{x_2} + \frac{\mu_1}{x_2+1} + \mu_\infty x_2\right] = 0.$$

This equation holds in $k(X \times X)$ and since the variables (x_1, Y_1) and (x_2, Y_2) are separated, each of the two terms equals zero. So we can drop the indices and we obtain an equation

$$Y^2 + Y = \mu_\infty x + \frac{\mu_0}{x} + \frac{\mu_1}{x+1} = \frac{\mu_\infty x^3 + \mu_\infty x^2 + (\mu_0+\mu_1)x + \mu_0}{x(x+1)}, \tag{3.3}$$

which has to be equivalent (after applying an automorphism of $\mathbb{A}^2_{\bar{K}}$) to the normal form (2.6) of the equation of X. The automorphism is given by i_S (see Section 2.4) with $S(x) = s \in \bar{K}$ satisfying $s^2 + s = \mu_1$. Hence (3.3) is equivalent to $Y^2 + Y = R(x)$ with $R(x) = \frac{\mu_\infty x^3 + (\mu_\infty + \mu_1)x^2 + \mu_0 x + \mu_0}{x(x+1)}$. Hence by uniqueness of the normal form, we obtain $a = \mu_\infty, b = \mu_1, c = \mu_0$ and therefore also the relations claimed in the proposition. $\square$

4. Degeneration of an ordinary genus 2 curve

The aim of this section is to describe (see formulae (4.10)) the degeneration of the theta basis of $H^0(JX, 2\Theta)$ when the ordinary curve X degenerates to a nonordinary one. This result will be central in the proof of Theorem 5.1 (Section 5).

For that purpose we denote by X/k a smooth curve with Hasse–Witt invariant equal to 1 and we introduce a family $\mathscr{X}$ over $R = k[[t]]$ such that the special fiber $\mathscr{X}_0$

is isomorphic to X and the generic fiber $\mathscr{X}_\eta$ is an ordinary genus 2 curve over $K = k((t))$. Here η (resp. 0) is the generic (resp. closed) point of $\mathrm{Spec}(R)$. Let $\mathscr{J}\mathscr{X}$ be its associated Jacobian scheme over $\mathrm{Spec}(R)$.

4.1. 2-divisible groups

Let $\mathscr{J}\mathscr{X}(2)$ be the 2-divisible group (see e.g. [D]) of $\mathscr{J}\mathscr{X}$, which is finite and flat over $\mathrm{Spec}(R)$. We consider the canonical exact sequence

$$0 \to \mathscr{J}\mathscr{X}(2)^0 \to \mathscr{J}\mathscr{X}(2) \to \mathscr{J}\mathscr{X}(2)^{\mathrm{et}} \to 0, \tag{4.1}$$

where $\mathscr{J}\mathscr{X}(2)^0$ (resp. $\mathscr{J}\mathscr{X}(2)^{\mathrm{et}}$) is a connected (resp. étale) 2-divisible group. Taking again the connected component of the Cartier dual $(\mathscr{J}\mathscr{X}(2)^0)^{\mathrm{D}}$ of $\mathscr{J}\mathscr{X}(2)^0$, we obtain the two inclusions

$$\mathscr{J}\mathscr{X}(2)^{00} \subset (\mathscr{J}\mathscr{X}(2)^0)^{\mathrm{D}}, \quad \mathscr{J}\mathscr{X}(2)^0 \subset \mathscr{J}\mathscr{X}(2),$$

with quotients given by the 2-divisible groups

$$\mathscr{J}\mathscr{X}(2)/\mathscr{J}\mathscr{X}(2)^0 = \mathscr{J}\mathscr{X}(2)^{\mathrm{et}} \cong \mathbb{Q}_2/\mathbb{Z}_2, \quad (\mathscr{J}\mathscr{X}(2)^0)^{\mathrm{D}}/\mathscr{J}\mathscr{X}(2)^{00} \cong \mathbb{G}_m(2).$$

The 2-divisible group $\mathscr{J}\mathscr{X}(2)^{00}$ is self-dual, of dimension 1 and of height 2. Because of the uniqueness of 2-divisible groups over k with these properties (see e.g. [D, Examples, p. 93]), the special fiber $\mathscr{J}\mathscr{X}(2)^{00}_0$ $(= \mathscr{J}\mathscr{X}(2)^{00} \otimes_R k)$ is isomorphic to the 2-divisible group associated to the supersingular elliptic curve E^{ss}/k. We recall that there exists a unique (up to isomorphism) supersingular curve E^{ss}, which is defined by $j = 0$. Therefore by a theorem of Serre-Tate [K], there exists an elliptic curve $\mathscr{E}_{\mathscr{X}}$ over $\mathrm{Spec}(R)$ such that $(\mathscr{E}_{\mathscr{X}})_0 \cong E^{\mathrm{ss}}$ and the associated 2-divisible group $\mathscr{E}_{\mathscr{X}}(2)$ is isomorphic to $\mathscr{J}\mathscr{X}(2)^{00}$ over $\mathrm{Spec}(R)$.

4.2. Degeneration of elliptic curves

In this section we compute the linear action of the 2-torsion point of $\mathscr{E}$ on the space of second-order theta functions $H^0(\mathscr{E}, 2\Theta)$ for a family of elliptic curves $\mathscr{E}/\mathrm{Spec}(R)$ with supersingular special fiber $\mathscr{E}_0 \cong E^{\mathrm{ss}}$ and ordinary generic fiber $\mathscr{E}_\eta$.

4.2.1. Addition on an ordinary elliptic curve

Let E be an ordinary elliptic curve defined over a field K of characteristic 2 by the homogeneous equation

$$Y^2Z + a_1TYZ = T^3 + a_2T^2Z + a_4TZ^2 \tag{4.2}$$

with $a_1, a_2, a_4 \in K$ and $a_1 \neq 0$. We take as origin the inflection point ∞ with projective coordinates $(0:1:0)$. The projection with center ∞ gives a $2:1$ morphism $E \xrightarrow{\pi} \mathbb{P}^1_K$, with T, Z projective coordinates on $\mathbb{P}^1_K$. The Abel–Jacobi map $E \to JE$, $e \mapsto \mathcal{O}_E(e - \infty)$ identifies E with JE. Under this identification the theta divisor Θ_B, associated to the canonical theta-characteristic $B = \mathcal{O}_E(P - \infty) \in JE[2]$, becomes the 2-torsion point P with projective coordinates $(0:0:1)$. Moreover, using this identification, we have $\mathcal{O}_{JE}(2\Theta) = \mathcal{O}_E(2P) = \pi^*\mathcal{O}_{\mathbb{P}^1}(1)$.

The point $B \in JE[2]$ induces a linear involution, denoted by g, on the space

$$W = H^0(JE, \mathcal{O}(2\Theta)) = H^0(E, \pi^*\mathcal{O}_{\mathbb{P}^1}(1)), \tag{4.3}$$

such that for all nonzero $s \in W$ we have $T_B \operatorname{div}(s) = \operatorname{div}(g.s)$. Here T_B denotes translation in JE by the point B. The space W has two distinguished bases: first the coordinate functions $\{T, Z\}$ and secondly the theta basis $\{T_0, T_1\}$ (see [LP, Section 2]). Since the canonical section $T_0 \in W$ (associated to the divisor Θ_B) is proportional to T, there exists a nonzero $a \in K$ such that

$$g.T = aZ, \quad g.Z = a^{-1}T.$$

In order to determine a in terms of the coefficients $a_i \in K$, we choose one of the two points on E with projective coordinates of the form $(1 : Y : 1)$ and call it A. By construction we have $A \in \operatorname{div}(T + Z)$ and, after applying T_B, we obtain $A + P \in T_B \operatorname{div}(T + Z)$. Since

$$T_B \operatorname{div}(T + Z) = \operatorname{div}(g.T + g.Z) = \operatorname{div}(aZ + a^{-1}T),$$

we deduce that

$$\left(\frac{T}{Z}\right)(A + P) = a^2.$$

Now the addition formula for elliptic curves (see e.g. [S, p. 59]) implies that $(\frac{T}{Z})(A + P) = a_4$. Hence $a = \sqrt{a_4}$ and the theta basis of W is given by

$$T_0 = T, \quad T_1 = g.T_0 = \sqrt{a_4}Z. \tag{4.4}$$

4.2.2. An example

We consider the family of elliptic curves $\mathscr{E}$ over $\operatorname{Spec}(R)$ defined by the homogeneous equation

$$V^2Z + t^4UVZ + VZ^2 = U^3 + UZ^2 \tag{4.5}$$

with origin $\infty = (0:1:0)$. The generic fiber $\mathscr{E}_\eta$ is an ordinary elliptic curve over $\operatorname{Spec}(K)$, with $K = k((t))$, and the special fiber is supersingular, i.e., $\mathscr{E}_0 \cong E^{ss}$. The

2-torsion point P_η of $\mathscr{E}_\eta$ has projective coordinates

$$P_\eta = t^6(u_0 : v_0 : 1),$$

with $u_0 = t^{-4}$, $v_0 = t^{-2} + t^{-6}$, which specializes to $\infty_0 \in \mathscr{E}_0$.

The R-module $H^0(\mathscr{E}, 2\Theta)$ is free, of rank 2, and an R-basis is given by $\{U, Z\}$. In order to compute the linear action g of the 2-torsion point $P \in \mathscr{E}$ on $H^0(\mathscr{E}, 2\Theta)$, we consider the generic fiber $\mathscr{E}_\eta$. The change of variables $T = U + u_0 Z$ and $Y = V + v_0 Z$ transforms Eq. (4.5) into (4.2), with $a_1 = t^4$, $a_2 = u_0$, and $a_4 = 1 + u_0^2 + t^4 v_0 = 1 + t^{-8} + t^2 + t^{-2}$. With the notation of Section 4.2.1 we find $a = t^{-4}(1 + t^3 + t^4 + t^5) = \sqrt{a_4}$. Therefore the action of g on $H^0(\mathscr{E}, 2\Theta)$ is given by the formulae

$$g.U = \frac{1}{1 + t^3 + t^4 + t^5} U + \frac{t^2 + t^4 + t^6}{1 + t^3 + t^4 + t^5} Z,$$

$$g.Z = \frac{t^4}{1 + t^3 + t^4 + t^5} U + \frac{1}{1 + t^3 + t^4 + t^5} Z,$$

and, using (4.4), the theta functions T_0 and T_1 can be expressed in the R-basis $\{U, Z\}$ as follows:

$$T_0 = t^4 U + Z,$$

$$T_1 = g.T_0 = (1 + t^3 + t^4 + t^5)Z.$$

Note that we multiplied both expressions by t^4, which will not affect the final calculations of Section 5 as we will work on the projectivization of the R-module $\mathscr{W}$. We also note that the two sections $T_0 \otimes_R k$ and $T_1 \otimes_R k$ coincide at the special fiber $H^0(\mathscr{E}_0, 2\Theta_{|\mathscr{E}_0}) \cong H^0(\mathscr{E}, 2\Theta) \otimes_R k$.

4.3. Specializing an ordinary curve

Let X/k be a smooth genus 2 curve with Hasse–Witt invariant equal to 1. Following, e.g. [L] X is birational to an affine curve given by an equation of the form

$$y^2 + xy = \lambda x^5 + \mu x^3 + x,$$

with $\lambda, \mu \in k$ and $\lambda \neq 0$. The projection $(x, y) \mapsto x$ defines a separable double cover $X \to \mathbb{P}^1_k$ ramified at 0 and ∞. Let $\mathbb{P}^1_R$ be the projective line over $R = k[[s]]$ with affine coordinate x. We introduce the family $\mathscr{X} \to \mathbb{P}^1_R$ defined by the projective closure of the affine curve with equation

$$y^2 + (sx^2 + x)y = \lambda x^5 + \mu x^3 + x.$$

The special fiber $\mathscr{X}_0/k$ equals X and the generic fiber $\mathscr{X}_\eta/K$ of the family $\mathscr{X}$ is a smooth ordinary curve of genus 2, which is birational to the curve (defined over a finite extension of K) given by the standard equation (2.5) with coefficients

$$a = \lambda/s^3, \quad b = \alpha^2 + \alpha, \quad c = s, \quad \text{and} \quad \alpha^2 = \lambda/s^3 + \mu/s + s. \tag{4.6}$$

Let $\mathscr{J}\mathscr{X}$ be the associated Jacobian scheme and $\mathscr{J}\mathscr{X}[2]/R$ be the group scheme of 2-torsion points. Then we have the following isomorphisms:

$$\mathscr{J}\mathscr{X}[2]_\eta \cong (\mathbb{Z}/2\mathbb{Z})^2 \times \mu_2^2/K, \quad \mathscr{J}\mathscr{X}[2]_0 \cong JX[2] \cong (\mathbb{Z}/2\mathbb{Z}) \times \mu_2 \times G_{1,1}/k,$$

where $G_{1,1}$ is the unique self-dual local–local group scheme of dimension 1 and length 4. Note that $G_{1,1}$ is isomorphic to the group scheme of 2-torsion points $E^{ss}[2]$ (see Section 4.2.2). The étale parts of both fibers can be described in terms of Weierstrass points as follows.

The 3 Weierstrass points of $\mathscr{X}_\eta \to \mathbb{P}^1_K$ are 0_η, ∞_η, and 1_η with affine coordinate $1/s$, which specialize to 0, ∞ and ∞, respectively. We obtain by (2.2) the three nonzero elements of $\mathscr{J}\mathscr{X}[2]^{\mathrm{et}}_\eta$, which we denote by $[0]_\eta$, $[1]_\eta$ and $[\infty]_\eta$. At the special fiber the nonzero 2-torsion point in $JX[2]^{\mathrm{et}} \cong \mathbb{Z}/2\mathbb{Z}$ equals $\mathrm{AJ}(0+\infty)$, which we denote by $[1]_0$. We see that $[1]_\eta$ and $[\infty]_\eta$ specialize to $[1]_0 \in JX[2]^{\mathrm{et}}$, and $[0]_\eta$ specializes to 0.

4.4. Decomposing Heisenberg groups

We are interested in the linear action of the Heisenberg group scheme $\mathscr{H}/R$, which is a central extension (see [M, p. 221])

$$0 \to \mu_2 \to \mathscr{H} \to \mathscr{J}\mathscr{X}[2] \to 0, \tag{4.7}$$

on the free R-module $\mathscr{W} = H^0(\mathscr{J}\mathscr{X}[2], 2\Theta)$ of rank 4. We choose the splitting over R of the connected-étale exact sequence

$$0 \to \mathscr{J}\mathscr{X}[2]^0 \to \mathscr{J}\mathscr{X}[2] \to \mathscr{J}\mathscr{X}[2]^{\mathrm{et}} \to 0$$

determined by the nonzero 2-torsion point $[1] = \mathrm{AJ}(0+\infty) \in \mathscr{J}\mathscr{X}[2]$. Note that $\mathscr{J}\mathscr{X}[2]^{\mathrm{et}} \cong \mathbb{Z}/2\mathbb{Z}$ and that $[\infty] \in \mathscr{J}\mathscr{X}[2]$ determines a different splitting. Passing to the Cartier dual we obtain a decomposition over R,

$$\mathscr{J}\mathscr{X}[2] = \mathbb{Z}/2\mathbb{Z} \times \mu_2 \times \mathscr{J}\mathscr{X}[2]^{00}.$$

Pulling back the central extension (4.7) by the canonical inclusions of $\mathbb{Z}/2\mathbb{Z} \times \mu_2$ and $\mathscr{J}\mathscr{X}[2]^{00}$ into $\mathscr{J}\mathscr{X}[2]$ we obtain the two Heisenberg groups $\mathscr{H}^{\mathrm{et}}$ and $\mathscr{H}^0$

$$0 \to \mu_2 \to \mathscr{H}^{\mathrm{et}} \to \mathbb{Z}/2\mathbb{Z} \times \mu_2 \to 0, \qquad 0 \to \mu_2 \to \mathscr{H}^0 \to \mathscr{J}\mathscr{X}[2]^{00} \to 0.$$

It is clear that the Heisenberg group scheme $\mathscr{H}$ (4.7) is isomorphic to the quotient $\mathscr{H}^0 \times \mathscr{H}^{\mathrm{et}}/\mu_2$, where μ_2 acts diagonally on $\mathscr{H}^0 \times \mathscr{H}^{\mathrm{et}}$. Let $\mathscr{W}^{\mathrm{et}}$ and $\mathscr{W}^0$ be the sub-R-modules of $\mathscr{W}$ fixed by the subgroups $\mathscr{H}^0$ and $\mathscr{H}^{\mathrm{et}}$ of $\mathscr{H}$. By general theory of Heisenberg groups, $\mathscr{W}$ is the unique irreducible $\mathscr{H}$-module of weight 1, which implies an $\mathscr{H}$-isomorphism

$$\mathscr{W} \cong \mathscr{W}^0 \otimes \mathscr{W}^{\mathrm{et}}.$$

Moreover $\mathscr{W}^0$ (resp. $\mathscr{W}^{\mathrm{et}}$) is the unique irreducible $\mathscr{H}^0$ (resp. $\mathscr{H}^{\mathrm{et}}$)-module of weight 1.

Let H/k be the Heisenberg group scheme associated to $\mathbb{Z}/2\mathbb{Z} \times \mu_2$ ($\cong E[2]$ for any ordinary elliptic curve E/k) and let W be the unique irreducible H-module of weight 1. Note that W is isomorphic (as H-module) to the space (4.3). It is clear that we have the following isomorphisms:

$$\mathscr{H}^{\mathrm{et}} \cong H \otimes_k R, \quad \mathscr{W}^{\mathrm{et}} \cong W \otimes_k R.$$

We will denote by $\{Z_0, Z_1\}$ the "constant" R-basis of $\mathscr{W}^{\mathrm{et}}$ induced by the theta basis $\{T_0, T_1\}$ of W, i.e., $Z_i \coloneqq T_i \otimes_k 1$.

The structure of the $\mathscr{H}^0$-module $\mathscr{W}^0$ is determined by analyzing the group scheme $\mathscr{J}\mathscr{X}[2]^{00}$. In Section 4.1 we considered the 2-divisible group $\mathscr{J}\mathscr{X}(2)^{00}$ and we showed the existence of an elliptic curve $\mathscr{E}_{\mathscr{X}}/R$ such that $\mathscr{E}_{\mathscr{X}}(2) \cong \mathscr{J}\mathscr{X}(2)^{00}$. In particular $\mathscr{E}_{\mathscr{X}}[2] \cong \mathscr{J}\mathscr{X}[2]^{00}$. We observe that the j-invariants of the elliptic curves $\mathscr{E}_{\mathscr{X}}/R$ and $\mathscr{E}/R$ (Section 4.2.2) lie in the maximal ideal of R, because $(\mathscr{E}_{\mathscr{X}})_0 \cong \mathscr{E}_0 \cong E^{\mathrm{ss}}$. Therefore, there exists a relation of the form

$$s^n = ut^m \tag{4.8}$$

between the two uniformizing parameters s in (4.6) and t in (4.5), with u invertible in $k[[t]]$ and $n, m \in \mathbb{N}^*$, and we can assume, after passing to the ramified cover given by (4.8), that the $\mathscr{H}^0$-module $\mathscr{W}^0$ equals $H^0(\mathscr{E}, 2\Theta)$.

In order to have a consistent notation we denote the R-basis $\{U, Z\}$ of $\mathscr{W}^0 = H^0(\mathscr{E}, 2\Theta)$ by $\{Z_0, Z_1\}$ and recall from Section 4.2.2 the transition formulae

$$T_0 = t^4 Z_0 + Z_1, \quad T_1 = (1 + t^3 + t^4 + t^5) Z_1. \tag{4.9}$$

Let $\{x_i\}$ and $\{z_i\}$ denote the dual K-bases of $\{T_i\}$ and $\{Z_i\}$ in both spaces $\mathscr{W}^0$ and $\mathscr{W}^{\mathrm{et}}$. Then the 4 tensors $z_{ij} \coloneqq z_i \otimes z_j \in \mathscr{W}^*$ form an R-basis and the dual theta functions $x_{ij} \coloneqq x_i \otimes x_j$ can be expressed as follows (after chasing denominators)

$$x_{00} = (1 + t^3 + t^4 + t^5) z_{00}, \quad x_{10} = z_{00} + t^4 z_{10},$$

$$x_{01} = (1 + t^3 + t^4 + t^5) z_{01}, \quad x_{11} = z_{01} + t^4 z_{11}. \tag{4.10}$$

Note that the coordinate x_{10} specializes to x_{00} and x_{11} specializes to x_{01}. Via the level 2 structure (2.3) this parallels the specialization of the 2-torsion points $[0]_\eta$, $[1]_\eta$, and $[\infty]_\eta$ (Section 4.3).

5. Equations of $\tilde{V}$ for nonordinary X in characteristic 2

It can be shown as in [LP] Section 5 that the identification $\mathrm{M}_X \to |2\Theta|$ extends to the relative case $\mathscr{X} \to \mathrm{Spec}(R)$, i.e., we have an isomorphism $\mathrm{M}_{\mathscr{X}} \to \mathbb{P}(\mathscr{W})$ over $\mathrm{Spec}(R)$. Therefore, the relative Frobenius morphism $\mathscr{X} \to \mathscr{X}_1$ (over $\mathrm{Spec}(R)$) induces by pull-back a rational map

$$\begin{array}{ccc} \mathbb{P}(\mathcal{W}_1) & \xrightarrow{\tilde{\mathcal{V}}} & \mathbb{P}(\mathcal{W}) \\ & \searrow \quad \swarrow & \\ & \mathrm{Spec}(R) & \end{array}$$

with $\mathscr{W}_1 = H^0(\mathscr{J}\mathscr{X}_1, 2\Theta_1)$. We recall that the map $\tilde{\mathscr{V}}$ is given by a linear system of 4 quadrics. Over the generic point $\eta \in \mathrm{Spec}(R)$ the equations of the map $\tilde{\mathscr{V}}_\eta : \mathbb{P}(\mathscr{W}_1)_\eta \to \mathbb{P}(\mathscr{W})_\eta$ are of the form [LP, Proposition 3.1(3)]

$$\tilde{\mathscr{V}} : x = (x_{ij}) \mapsto (\lambda_{00}P_{00}(x) : \lambda_{01}P_{01}(x) : \lambda_{10}P_{10}(x) : \lambda_{11}P_{11}(x)) \tag{5.1}$$

with

$$P_{00}(x) = x_{00}^2 + x_{01}^2 + x_{10}^2 + x_{11}^2, \quad P_{10}(x) = x_{00}x_{10} + x_{01}x_{11},$$

$$P_{01}(x) = x_{00}x_{01} + x_{10}x_{11}, \quad P_{11}(x) = x_{00}x_{11} + x_{10}x_{01}.$$

Here we use the K-basis of theta coordinates x_{ij} on $\mathbb{P}(\mathscr{W}_1)_\eta$ and $\mathbb{P}(\mathscr{W})_\eta$. Moreover Proposition 3.1 relates the coefficients λ_{ij} (defined up to a scalar) to the coefficients $a, b, c \in K$ (4.6),

$$(\lambda_{00} : \lambda_{01} : \lambda_{10} : \lambda_{11}) = (\sqrt{abc} : \sqrt{b} : \sqrt{c} : \sqrt{a}).$$

Since the theta coordinates x_{ij} are no longer independent after specialization, we express Eq. (5.1) of $\tilde{\mathscr{V}}$ in the R-basis $\{z_{ij}\}$ using transition formulae (4.10). A straightforward computation shows that the map

$$\tilde{\mathscr{V}} : z = (z_{ij}) \mapsto (R_{00}(z) : R_{01}(z) : R_{10}(z) : R_{11}(z)) \tag{5.2}$$

is given by the quadrics

$$R_{00}(z) = \frac{\sqrt{abc}}{1 + t^3 + t^4 + t^5}[(t^{12} + t^{16} + t^{20})(z_{00}^2 + z_{01}^2) + t^{16}(z_{10}^2 + z_{11}^2)],$$

$$R_{01}(z) = \frac{\sqrt{b}}{1 + t^3 + t^4 + t^5}[(t^{12} + t^{16} + t^{20})z_{00}z_{01} + t^8(z_{00}z_{11} + z_{10}z_{01}) + t^{16}z_{10}z_{11}],$$

$$R_{10}(z) = \frac{1}{t^4}[R_{00} + \sqrt{c}(1 + t^6 + t^8 + t^{10})(z_{00}^2 + z_{01}^2 + t^8(z_{00}z_{10} + z_{11}z_{01}))],$$

$$R_{11}(z) = \frac{1}{t^4}[R_{01} + \sqrt{a}(1 + t^6 + t^8 + t^{10})t^8(z_{00}z_{11} + z_{01}z_{10})]. \tag{5.3}$$

Note that we square the coefficients in (4.10) when considering coordinates on $\mathbb{P}(\mathscr{W}_1)$. At the special fiber map (5.2) specializes to $\tilde{\mathscr{V}}_0$ obtained by putting $t = 0$ after having divided the R_{ij}'s by t^α, where α is the lowest valuation appearing in expressions (5.3). The map $\tilde{\mathscr{V}}_0$ coincides with the Verschiebung $\tilde{V}$ of the curve X, because both maps extend to rational maps over R and coincide over K. Since the image of $\tilde{V}$ is nondegenerate (it contains the Kummer surface $\mathrm{Kum}_X \subset |2\Theta|$), the lowest valuations for each of the 4 quadrics R_{ij} coincide (otherwise the image of $\tilde{\mathscr{V}}_0$ is contained in a hyperplane).

We work out the specialization of the quadrics as follows. We write $v = \frac{m}{n}$ and replace s by vt^v, with $v \in k[[t]]$ invertible, in the expression of the coefficients a, b, c of (4.6). Note that the (rational) valuations of $\sqrt{a}, \sqrt{b}, \sqrt{c}$ are $-\frac{3}{2}v, -\frac{3}{2}v, \frac{1}{2}v$, respectively. First we observe that the valuations of R_{01} and R_{00} equal $8 - \frac{3}{2}v$ and $12 - \frac{5}{2}v$, respectively. Since they coincide, we obtain $v = 4$, i.e.,

$$R_{00} = t^2(z_{00}^2 + z_{01}^2) + \text{h.o.t.}, \quad R_{01} = t^2(z_{00}z_{11} + z_{10}z_{01}) + \text{h.o.t.},$$

up to some multiplicative nonzero constants. Next we see that the expansions of R_{10} and R_{11} are given by

$$R_{10} = t^2(z_{00}^2 + z_{01}^2 + z_{10}^2 + z_{11}^2) + \text{h.o.t.}, \quad R_{11} = t^2(z_{00}z_{01}) + \text{h.o.t.},$$

up to some multiplicative nonzero constants and some multiple of R_{00} and R_{01}, respectively. Thus we have shown

Theorem 5.1. *Let X be a smooth curve with Hasse–Witt invariant equal to* 1. *There exist coordinates $\{z_{ij}\}$ on $|2\Theta_1|$ and $\{y_{ij}\}$ on $|2\Theta|$ such that the equations of $\tilde{V}$ are given by*

$$|2\Theta_1| \xrightarrow{\tilde{V}} |2\Theta|, \quad z = (z_{ij}) \mapsto y = (y_{ij}) = (\lambda_{00}Q_{00}(z) : \lambda_{01}Q_{01}(z) : \lambda_{10}Q_{10}(z) : \lambda_{11}Q_{11}(z))$$

with

$$\begin{aligned} Q_{00}(z) &= z_{00}^2 + z_{01}^2, & Q_{10}(z) &= z_{00}^2 + z_{01}^2 + z_{10}^2 + z_{11}^2, \\ Q_{01}(z) &= z_{00}z_{11} + z_{10}z_{01}, & Q_{11}(z) &= z_{00}z_{01}, \end{aligned}$$

and the λ_{ij}'s are nonzero constants depending on the curve X.

Remark 5.2. We note that the equations of $\tilde{V}$ given in Theorem 5.1 are written in two sets of coordinates on $|2\Theta|$ and $|2\Theta_1|$ which do not necessarily correspond under the k-semi-linear isomorphism $JX_1 \to JX$.

Remark 5.3. In case X is a nonordinary curve with Hasse–Witt invariant equal to 0, i.e., X is supersingular, we observe that the 2-divisible group $JX(2) = JX(2)^{00}$ (see Section 4.1) is self-dual, of dimension 2 and height 4. There exists a finite number of isomorphism classes of such 2-divisible groups over k (see [D, p. 93]). Moreover one can show that $JX(2)$ cannot be isomorphic to the product $E^{ss}(2) \times E^{ss}(2)$.

As in [LP, Section 6] we can easily deduce from Theorem 5.1 a full description of the Verschiebung $\tilde{V}$. Since the computations are straightforward and similar to those of [LP, Proposition 6.1], we leave them to the reader.

Proposition 5.4. *Let X be a smooth genus 2 curve with Hasse–Witt invariant equal to* 1.

1. *There exists a unique stable bundle $E_{BAD} \in \mathrm{M}_{X_1}$, which is destabilized by the Frobenius map, i.e., $F^* E_{BAD}$ is not semi-stable. We have $E_{BAD} = F_* B^{-1}$ and its projective coordinates are* $(0:0:1:1)$.
2. *Let H_1 be the hyperplane in $|2\Theta_1|$ defined by $z_{00} + z_{01} = 0$. The map $\tilde{V}$ contracts H_1 to the conic $\mathrm{Kum}_X \cap H$, where H is the hyperplane in $|2\Theta|$ defined by $y_{00} = 0$.*
3. *The fiber of $\tilde{V}$ over a point $[E] \in \mathrm{M}_X$ is*
   - ○ *a nondegenerate $\mathbb{Z}/2\mathbb{Z}$-orbit of a point $[E_1] \in \mathrm{M}_{X_1}$, if $[E] \notin H$,*
   - ○ *empty, if $[E] \in H \backslash (H \cap \mathrm{Kum}_X)$,*
   - ○ *a projective line passing through E_{BAD}, if $[E] \in H \cap \mathrm{Kum}_X$.*

   In particular, $\tilde{V}$ is dominant and nonsurjective. The separable degree of $\tilde{V}$ is 2.

6. Equations of $\tilde{V}$ in characteristic 3

Let X be a smooth curve of genus 2 defined over a field of characteristic 3. The main result of this section is

Theorem 6.1. 1. *There exists an embedding $\alpha: \mathrm{Kum}_X \hookrightarrow |2\Theta_1|$ such that the equality of divisors in $|2\Theta_1|$*

$$\tilde{V}^{-1}(\mathrm{Kum}_X) = \mathrm{Kum}_{X_1} + 2\alpha(\mathrm{Kum}_X)$$

holds scheme-theoretically.

2. *The cubic equations of $\tilde{V}$ are given by the* 4 *partials of the quartic equation of the Kummer surface $\alpha(\mathrm{Kum}_X) \subset |2\Theta_1|$. In other words, $\tilde{V}$ is the polar map of the surface* $\alpha(\mathrm{Kum}_X)$.

Proof. Given $L \in JX$ and a section $\varphi \in H^0(X, \omega L^2)$, we can consider, using adjunction and relative duality for the map F, the homomorphism

$$F_* L \xrightarrow{\varphi} L\omega,$$

where we also write L for the pull-back $\imath^*L$ under the k-semi-linear isomorphism $\imath : X_1 \to X$. The map φ is surjective as is seen as follows: suppose φ vanishes at $x \in X$, then, again by adjunction and relative duality, we obtain a nonzero map $F^*(L^{-1}\omega^{-1}(x)) = L^{-3}\omega^{-3}(3x) \to L^{-1}\omega^{-2}$, which is impossible for degree reasons.

We define $E_L := \ker(\varphi)$. Hence there is an exact sequence of vector bundles over X_1

$$0 \to E_L \to F_*L \xrightarrow{\varphi} L\omega \to 0. \tag{6.1}$$

We have the following.

Lemma 6.2. 1. *For any* $L \in JX$, *the bundle* E_L *is semi-stable and* $\det E_L = \mathcal{O}_{X_1}$.

2. *If* L *is of the form* $\kappa^{-1}(x)$, *for* κ *a theta-characteristic and* $x \in X$, *then* $E_L \cong B \otimes \kappa^{-1}$ *and* F^*E_L *is not semi-stable. Otherwise* F^*E_L *is strictly semi-stable and* S*-equivalent to* $L \oplus L^{-1}$.

Proof. (1) Let M be a line subbundle of $E_L \hookrightarrow F_*L$. By adjunction we obtain a nonzero map $F^*M \to L$, hence $\deg M \leqslant 0$ and E_L is semi-stable. By (6.1) and since $\det F_*L = \det(F_*\mathcal{O}_X) \otimes L = \det B \otimes L = \omega L$, we obtain $\det E_L = \mathcal{O}_{X_1}$. Here B denotes the sheaf of locally exact differential forms of X_1 (see [R, Section 4.1]).

(2) By adjunction we obtain a nonzero map $F^*E_L \xrightarrow{\pi} L$. Suppose that π vanishes at one point $x \in X$, i.e., π factorizes through a nowhere vanishing map $F^*E_L \to L(-x)$. Again by adjunction we obtain an injective map $E_L \to F_*(L(-x))$ with quotient map $\alpha : F_*(L(-x)) \to L\omega(-x)$. But α corresponds via adjunction and relative duality to a nonzero global section of $\omega L^2(-2x)$, i.e., L is of the form $\kappa^{-1}(x)$. It easily follows that $E_L \cong B \otimes \kappa^{-1}$. Similarly one shows that π cannot vanish at more than one point and it follows that, if L is not of the form $\kappa^{-1}(x)$, $F^*E_L \to L$ is nowhere vanishing and F^*E_L is S-equivalent to $L \oplus L^{-1}$. $\square$

We observe that for L such that $L^2 \neq \mathcal{O}$, $\dim H^0(\omega L^2) = 1$, hence φ is uniquely defined (up to a scalar). For L such that $L^2 = \mathcal{O}$, we obtain a projective line $\mathbb{P}^1_L$ of rank 2 vector bundles $E_{L,\varphi}$ with $\varphi \in |\omega|$. The variety of pairs (L, φ) is isomorphic to the blowing-up $\mathrm{Bl}_2(JX)$ of JX at the 16 2-torsion points $L \in JX[2]$. Hence we obtain a morphism defined on the k-valued point (L, φ) by

$$e : \mathrm{Bl}_2(JX) \to \mathrm{M}_{X_1} \cong |2\Theta_1|, \quad (L, \varphi) \mapsto E_{L,\varphi}.$$

The existence of the morphism e follows from the coarse moduli property of M_{X_1} and the existence of a family $\mathscr{E}$ over $X \times \mathrm{Bl}_2(JX)$ such that $\mathscr{E}_{|X \times \{(L,\varphi)\}} \cong E_{L,\varphi}$. In order to construct the family $\mathscr{E}$ we consider the Abel–Jacobi map $\mathrm{AJ} : \mathrm{Sym}^2(X) \to \mathrm{Pic}^2(X)$ and observe that $\mathrm{Bl}_2(JX)$ is the fiber product of $\mathrm{Sym}^2(X)$ under the base change $JX \to \mathrm{Pic}^2(X), L \mapsto \omega L^2$. We consider the incidence divisor $\Delta \subset X \times \mathrm{Sym}^2(X)$ and pull-back its associated section to $X \times \mathrm{Bl}_2(JX)$. Using relative duality and adjunction, we obtain (exactly as over X) the family $\mathscr{E}$. Since we will not use the exact definition of $\mathscr{E}$, we leave the details to the reader.

Now we will determine the image of the morphism e. We consider the set of semi-stable bundles

$$\tilde{V}^{-1}(\mathrm{Kum}_X) = \{E \in \mathrm{M}_{X_1} \mid F^* E \text{ strictly semi-stable}\}.$$

Lemma 6.3. *The hypersurface* $\tilde{V}^{-1}(\mathrm{Kum}_X)$ *has two irreducible components: the Kummer surface* Kum_{X_1} *and the image* $e(\mathrm{Bl}_2(JX))$.

Proof. For any stable $E \in \tilde{V}^{-1}(\mathrm{Kum}_X)$ there exists a nonzero map $F^*E \to L$ for some $L \in JX$. By adjunction we obtain a nonzero map $E \xrightarrow{i} F_* L$. Suppose that i has not maximal rank, i.e., i factorizes through $E \to M \to F_* L$, for some line bundle M. Then, on one hand, stability of E implies that $\deg M > 0$ and, on the other hand, we have a nonzero map $F^* M \to L$, which implies $\deg M \leqslant 0$, a contradiction. Hence i is generically injective. Suppose that i drops rank at x. Then there exists a quotient map $F_* L \to \omega L(-x)$, which corresponds via adjunction and relative duality to a nonzero global section of $\omega L^3(-3x)$, a contradiction. Hence E equals E_L. $\square$

By Proposition 7.2 the map $\tilde{V}$ is given by a linear system $|\mathscr{L}|$ of 4 cubics on $|2\Theta_1|$. The key fact underlying Theorem 6.1 is a striking relationship between cubics and quartics on $|2\Theta_1|$ [vG, Proposition 2]: the 4 cubics in $|\mathscr{L}|$ are the 4 partial derivatives of a Heisenberg invariant quartic, whose zero divisor we denote by $Q \subset |2\Theta_1|$. By [R, Remark 4.1.2(2)] the 16 points $B \otimes \kappa^{-1}$ (a $JX_1[2]$-orbit) are contained in the base locus of $|\mathscr{L}|$, so that the quartic Q is singular at the 16 points $B \otimes \kappa^{-1}$. Hence Q is a Kummer surface with polar map $\tilde{V}$ [GD, Theorem 2.20 and Lemma 2.19]. We will show that

$$Q \cong \mathrm{Kum}_X \quad \text{and} \quad Q = e(\mathrm{Bl}_2(JX)).$$

The 16 nodes of the quartic Q form a so-called 16_6-configuration [GD, Section 1]: there are 16 hyperplanes containing each 6 nodes, which moreover lie on a conic. In order to determine the image $\tilde{V}(Q)$, which will be isomorphic to Q—since Q is self-dual [GD, Corollary 4.27]—it will be enough to determine the images of these 16 conics. Indeed the 16 conics will be contracted by $\tilde{V}$ to the 16 nodes of $\tilde{V}(Q)$ [GD, Remark 4.29].

We now give an additional description of these 16 conics through the 16 6-tuples of nodes.

Lemma 6.4. *For any* $L \in JX[2]$:

1. *the image* $e(\mathbb{P}^1_L)$ *in* M_{X_1} *is a conic,*
2. *for a Weierstrass point* $w \in X$, *if* $\mathrm{Div}(\varphi) = 2w$, *we have* $e(\varphi) \cong B \otimes L^{-1}(-w)$.

Proof. (1) Since e is $JX[2]$-equivariant, it will be enough to prove the statement for $L = \mathcal{O}$. We have a family $\{E_\varphi\}$ of semi-stable rank 2 vector bundles parameterized by $\mathbb{P}^1 = |\omega|$ and defined (pointwise) by the exact sequence

$$0 \to E_\varphi \to F_*\mathcal{O} \xrightarrow{\varphi} \omega \to 0. \tag{6.2}$$

Let us define the family $\{E_\varphi\}$: we denote by p and q the projections of $X \times \mathbb{P}^1$ on the factors X and $\mathbb{P}^1$. Then we have a "universal" section over $X \times \mathbb{P}^1$

$$q^*\mathcal{O}_{\mathbb{P}^1}(-1) \to p^*\omega.$$

Now we tensorize with $p^*\omega^{-3} \otimes q^*\mathcal{O}(1)$ and we obtain

$$(\mathrm{Id} \times F)^* p^*\omega^{-1} = p^*\omega^{-3} \to p^*\omega^{-2} \otimes q^*\mathcal{O}(1),$$

which, by adjunction, transforms into a map

$$p^*\omega^{-1} \to (\mathrm{Id} \times F)_*(p^*\omega^{-2} \otimes q^*\mathcal{O}(1)) = p^*F_*\omega^{-2} \otimes q^*\mathcal{O}(1).$$

Taking the dual and using relative duality $F_*\omega^{-2} = (F_*\mathcal{O})^\vee$ we define the family $\mathscr{E}$ over $X \times \mathbb{P}^1$ as the kernel (again we check surjectivity as above)

$$0 \to \mathscr{E} \to p^*F_*\mathcal{O} \otimes q^*\mathcal{O}(-1) \to p^*\omega \to 0. \tag{6.3}$$

By construction we have $\mathscr{E}_{|X \times \{\varphi\}} \cong E_\varphi$. We consider a hyperplane in M_{X_1} of the form

$$H = \{E \in \mathrm{M}_{X_1} \mid \dim H^0(X_1, E \otimes M) > 0\},$$

with $M \in \mathrm{Pic}^1(X_1)$. We recall that H is defined as a determinant divisor and by functoriality of determinant divisors, the pull-back divisor $e^*(H)$ (we also denote by e the restricted morphism $e : \mathbb{P}^1 \to \mathrm{M}_{X_1}$) equals the determinant divisor of the vector bundle map

$$q_*(p^*(F_*\mathcal{O} \otimes M) \otimes q^*\mathcal{O}(-1)) = H^0(M^3) \otimes \mathcal{O}(-1) \to q_*(p^*\omega M) = H^0(\omega M),$$

obtained by tensorizing (6.3) with p^*M and taking direct image under q. Note that the higher direct images R^1q_* are zero. Since $h^0(M^3) = h^0(\omega M) = 2$, we obtain $\deg e^*(H) = 2$.

(2) We tensorize the exact sequence (6.1) with L, using $F_*L \otimes L = F_*(L \otimes F^*L) = F_*\mathcal{O}_X$ (projection formula)

$$\begin{array}{ccccccccc} & & & & \mathcal{O}_{X_1} & & & & \\ & & & & \downarrow & \searrow^{\varphi} & & & \\ 0 & \longrightarrow & E_{L,\varphi} \otimes L & \longrightarrow & F_*\mathcal{O}_X & \longrightarrow & \omega_{X_1} & \longrightarrow & 0 \\ & & & \searrow^{\hat{\varphi}} & \downarrow & & & & \\ & & & & B & & & & \end{array}$$

The vertical arrows form an exact sequence (see [R, Section 4.1]). The upper diagonal map defined as composite map $\mathcal{O}_{X_1} \to F_* \mathcal{O}_X \to \omega_{X_1}$ equals φ, which implies that the lower diagonal map $\hat{\varphi} : E_{L,\varphi} \otimes L \to B$ vanishes at w. Hence, using stability of B and $\det B = \omega$, we obtain $E_{L,\varphi} \otimes L \cong B(-w)$. $\square$

It immediately follows from this lemma that the 16 conics of the 16_6-configuration are the conics $e(\mathbb{P}^1_L)$, with $L \in JX[2]$. It is clear by construction that the conic $e(\mathbb{P}^1_L)$ is contracted by $\tilde{V}$ to the point $[L \oplus L^{-1}] \in \mathrm{M}_X$. Hence the image $\tilde{V}(Q)$ is a Kummer surface singular along the 16 points $[L \oplus L^{-1}]$, with $L \in JX[2]$. Since a Kummer surface is uniquely defined by its 16 nodes [GD, Lemma 2.19], we obtain $\tilde{V}(Q) = \mathrm{Kum}_X$ and by self-duality of Q, we conclude that $Q \cong \mathrm{Kum}_X$.

Finally we also have obtained that $Q \subset \tilde{V}^{-1}(\mathrm{Kum}_X)$. Obviously $Q \neq \mathrm{Kum}_{X_1}$, so $Q = e(\mathrm{Bl}_2(JX))$, since both surfaces are irreducible. This completes the proof of Theorem 6.1. $\square$

Remark 6.5. The rational map $e : \mathrm{Kum}_X \to \mathrm{M}_{X_1}$ (defined away from the 16 nodes of Kum_X) is the birational inverse of $\tilde{V}$.

Corollary 6.6. 1. *The map $\tilde{V}$ has exactly* 16 *base points, which correspond bijectively to the*

- 16 *nodes of the surface* $\alpha(\mathrm{Kum}_X) \subset |2\Theta_1|$,
- 16 *stable rank* 2 *vector bundles* $B \otimes \kappa^{-1} \in \mathrm{M}_{X_1} \cong |2\Theta_1|$, *where B is the bundle of locally exact differentials and κ a theta-characteristic of X_1.*

2. *The map $\tilde{V}$ is surjective, separable and of degree* 11.

Proof. We only have to show part 2, since part 1 is clear from the proof of Theorem 6.1. We recall that the rational map $\tilde{V}$, which is defined away from the 16 points $B \otimes \kappa^{-1}$, coincides with the polar map of the Kummer surface Kum_X. It is well-known that $\tilde{V}$ can be resolved into a morphism $\mathscr{V} : \mathrm{Bl}(|2\Theta_1|) \to |2\Theta|$ by blowing-up these 16 points in $|2\Theta_1|$. We denote by E_κ the exceptional divisor over $B \otimes \kappa^{-1}$ and by $H_\kappa \subset |2\Theta|$ the hyperplane $\mathscr{V}(E_\kappa)$. Note that the H_κ are the 16 tropes of $\mathrm{Kum}_X \subset |2\Theta|$ and that $\mathscr{V}_{|E_\kappa}$ is a linear isomorphism. It is clear that the image of $\tilde{V}$ contains the complement of the 16 hyperplanes H_κ.

Let us check that the H_κ are also contained in the image of $\tilde{V}$: a simple computation shows that the cubic $C_\kappa := \tilde{V}^{-1}(H_\kappa) \subset |2\Theta_1|$ is singular at the point $B \otimes \kappa^{-1}$ and that the restriction of $\tilde{V}$ to the cubic C_κ coincides with the (birational) projection with center $B \otimes \kappa^{-1}$. Moreover, the projectivized tangent cone at $B \otimes \kappa^{-1}$ to C_κ is the conic $Q_\kappa \subset H_\kappa$ through the 6 nodes (recall that $2Q_\kappa = H_\kappa \cap \mathrm{Kum}_X$). Therefore any point in $H_\kappa \backslash Q_\kappa$ lies in the image of $\tilde{V}$. To finish the argument we observe that $Q_\kappa \subset \mathrm{Kum}_X$ and that $\tilde{V} : \mathrm{Kum}_{X_1} \to \mathrm{Kum}_X$ is surjective.

Since $\tilde{V}$ is given by 4 cubic equations, whose scheme-theoretical base locus is the reduced set of 16 points $B \otimes \kappa^{-1} \in \mathrm{M}_{X_1}$, we obtain

$\deg \tilde{V} = 3^3 - 16 = 11$. Since the characteristic $p = 3$ does not divide $\deg \tilde{V}$, $\tilde{V}$ is separable. $\square$

Remark 6.7. 1. We recall [LP, Remark 6.2] that surjectivity only holds for S-equivalence classes (not isomorphism classes!). In fact, there always exist semi-stable bundles E which do not descend by Frobenius.

2. The number of base points and the degree of $\tilde{V}$ was also obtained in [O] by computing the number of connections (on certain unstable bundles) with zero p-curvature.

3. It would be interesting to have an explicit description of the 11 vector bundles in a general fiber $\tilde{V}^{-1}(E)$ of the polar map $\tilde{V}$.

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Appendix A. Base points of V

In this section we consider a smooth curve X of genus $g \geqslant 2$ defined over an algebraically closed field k of characteristic $p > 0$. We denote by $\mathrm{M}_X(r)$ (resp. $\mathrm{M}_{X_1}(r)$) the moduli space of semi-stable rank r vector bundles over X (resp. X_1) with fixed trivial determinant and by $\mathscr{L}$ (resp. $\mathscr{L}_1$) the determinant line bundle [KM] over $\mathrm{M}_X(r)$ (resp. $\mathrm{M}_{X_1}(r)$). The relative Frobenius map $F: X \to X_1$ induces by pull-back a rational map

$$V: \ \mathrm{M}_{X_1}(r) \to \mathrm{M}_X(r),$$

called the Verschiebung. Let $\mathscr{I}$ be the indeterminacy locus of V, i.e., the closed subscheme of $\mathrm{M}_{X_1}(r)$ consisting of semi-stable bundles E such that F^*E is not semi-stable. Let $U = \mathrm{M}_{X_1}(r) \backslash \mathscr{I}$ be the open subset where V is a morphism.

A.1. General facts

Proposition A.1. *We have an isomorphism* $V^*(\mathscr{L}) \cong (\mathscr{L}_1^{\otimes p})_{|U}$.

Proof. Let $\mathscr{M}_X(r)$ and $\mathscr{M}_{X_1}(r)$ be the moduli stacks of rank r vector bundles over X and X_1 and let $\mathscr{E}$ and $\mathscr{E}_1$ be the universal bundles with trivialized determinant on $X \times \mathscr{M}_X(r)$ and $X_1 \times \mathscr{M}_{X_1}(r)$. It is well-known that the inverses of the determinant of cohomology, which we denote by $\det Rp_*\mathscr{E}$ and $\det Rp_{1*}\mathscr{E}_1$ descend (after restriction to the semi-stable loci) to the line bundles $\mathscr{L}$ and $\mathscr{L}_1$ on the moduli spaces $\mathrm{M}_X(r)$ and $\mathrm{M}_{X_1}(r)$. Now, since $\det Rp_*$ commutes with base change, we have an isomorphism over the moduli stack $\mathscr{M}_{X_1}(r)$

$$V^*(\det Rp_*\mathscr{E}) \cong \det Rp_*((F \times \mathrm{Id})^*\mathscr{E}_1).$$

Moreover, we have a commutative diagram

$$\begin{array}{ccc} X \times \mathcal{M}_{X_1}(r) & \xrightarrow{F \times \mathrm{Id}} & X_1 \times \mathcal{M}_{X_1}(r) \\ & \searrow^{p} \quad \swarrow^{p_1} & \\ & \mathcal{M}_{X_1}(r) & \end{array}$$

where p and p_1 denote the projections on the second factor. Since $F \times \mathrm{Id}$ is an affine morphism, we have $R^1(F \times \mathrm{Id})_* = 0$. Hence

$$\det Rp_*((F \times \mathrm{Id})^*\mathscr{E}_1) \cong \det Rp_{1*}((F \times \mathrm{Id})_*(F \times \mathrm{Id})^*\mathscr{E}_1) \cong \det Rp_{1*}(\mathscr{E}_1 \boxtimes F_*\mathcal{O}_X).$$

The last equality follows from the projection formula. Using a filtration by line bundles of the rank p bundle $F_*\mathcal{O}_X$ and by showing that $\det Rp_{1*}(\mathscr{E}_1 \boxtimes \mathcal{O}_{X_1}(D)) = \det Rp_{1*}(\mathscr{E}_1)$ for an effective divisor D—here we use the fact that $\det \mathscr{E}_1$ is trivialized—we show that $\det Rp_{1*}(\mathscr{E}_1 \boxtimes F_*\mathcal{O}_X) \cong (\det Rp_{1*}\mathscr{E}_1)^{\otimes p}$. We obtain the isomorphism of the lemma by descent on U. $\square$

Proposition A.2. *If $g = 2$ and $r = 2$, then* $\dim \mathscr{I} = 0$ *and the rational map $\tilde{V}$ is given by polynomials of degree p.*

Proof. The fact that $\dim \mathscr{I} = 0$ is proved in Theorem 3.2 [JX]. This implies that $V^*(\mathscr{L})$ extends uniquely to $\mathscr{L}_1^{\otimes p}$ over M_{X_1} and the lemma follows from the isomorphism $\mathscr{L}_1 \cong \mathcal{O}_{\mathbb{P}^3}(1)$. $\square$

Remark A.3. For general g, r, p we do not know an estimate of the dimension of $\mathscr{I}$. For $r = 2$ and $p = 2$ see [JRXY].

A.2. Existence of base points

Theorem A.4. *The indeterminacy locus $\mathscr{I}$ is nonempty.*

Proof. Firstly, it will be enough to show nonemptiness of $\mathscr{I}$ in the case $r = 2$, since taking direct sums with the trivial bundle implies nonemptiness for arbitrary r. Secondly, it suffices to show nonemptiness of $\mathscr{I}$ after a field extension k'/k, with k' algebraically closed.

Let $\overline{\mathrm{M}_g}$ be the coarse moduli space of stable genus g curves defined over k, which is an irreducible projective variety [DM]. Let η be the generic point of $\overline{\mathrm{M}_g}$. The choice of a geometric point $\bar{\eta}$ over η defines a smooth curve $\mathscr{X}_{\bar{\eta}}$ over $\overline{k(\eta)}$, the algebraic closure of the function field $k(\eta)$ of $\overline{\mathrm{M}_g}$. The curve $\mathscr{X}_{\bar{\eta}}$ is defined over a finite extension K of $k(\eta)$ and we denote by $\mathscr{X}_K$ some model of $\mathscr{X}_{\bar{\eta}}$, i.e., $\mathscr{X}_K \times_K \overline{k(\eta)} \cong \mathscr{X}_{\bar{\eta}}$.

The curve X/k defines a k-rational point x of $\overline{\mathrm{M}_g}$, which lies in the closure of η. The local ring A_x at the generic point of the exceptional divisor of the blowing-up of $\overline{\mathrm{M}_g}$ at the point x is a discrete valuation ring with fraction field $k(\eta)$ and residue field containing k. By the stable reduction theorem [DM, Corollary 2.7] there exists a finite extension L of K, and therefore also of $k(\eta)$, such that $\mathscr{X}_L$ is the generic fiber of a stable curve $\mathscr{X}$ over the integral closure A of A_x in L. Note that A is a discrete valuation ring with fraction field L and with residue field, denoted by $k(s)$, containing k. Moreover the diagram

$$\begin{tikzcd} \mathrm{Spec}(A) \arrow[d] \arrow[dr, "\mathscr{X}"] & \\ \mathrm{Spec}(A_x) \arrow[r] & \overline{\mathrm{M}_g} \end{tikzcd}$$

commutes when restricted to $\mathrm{Spec}(L) \hookrightarrow \mathrm{Spec}(A)$ and therefore commutes because $\overline{\mathrm{M}_g}$ is separated. It follows that the special point $s \in \mathrm{Spec}(A)$ maps to x, i.e., there exists an isomorphism $X \times_k \overline{k(s)} \cong \mathscr{X}_s \times_{k(s)} \overline{k(s)}$.

In summary, we have constructed a stable curve $\mathscr{X}$ over a discrete valuation ring A with generic fiber $\mathscr{X}_L$ and geometric special fiber isomorphic to $X \times_k \overline{k(s)}$. The fraction field of A is L and its residue field $k(s)$.

We now choose a nodal genus g-curve which is a union of $\mathbb{P}^1_k$'s meeting transversally, denoted by X', defining a closed point x' in the boundary of $\overline{\mathrm{M}_g}$. Repeating the above construction with x' instead of x, we obtain a stable curve $\mathscr{X}'$ over a discrete valuation ring A' with generic fiber $\mathscr{X}_{L'}$ and geometric special fiber isomorphic to $X' \times_k \overline{k(s')}$. The fraction field of A' is L', a finite extension of $k(\eta)$, and its residue field is $k(s')$. Moreover the isomorphism $X' \times_k \overline{k(s')} \cong \mathscr{X}_{s'} \times_{k(s')} \overline{k(s')}$ is defined over a finite extension of $k(s')$.

We choose a finite extension of $k(\eta)$ containing both L and L', which we call again L, and take the integral closures in L of A and A', which we call again A and A'. Thus we have constructed two stable curves $\mathscr{X}$ and $\mathscr{X}'$ over A and A' such that $\mathscr{X}_L \cong \mathscr{X}'_L$ and which specialize to X and X' respectively.

Let $\hat{L}$ be the fraction field of the completion $\hat{A}'$ of A'. By construction the curve $\mathscr{X}_{\hat{L}} \cong \mathscr{X}'_{\hat{L}}$ is a Mumford-Tate curve and, by the main result of [G], there exists a stable rank 2 vector bundle $\hat{\mathscr{E}}$ over $\mathscr{X}_{\hat{L}}$ such that $F^*\hat{\mathscr{E}}$ is not semi-stable.

Lemma A.5. *There exists a finite extension L_1 of L contained in the field $\hat{L}$ and a stable bundle $\mathscr{E}_1$ over $\mathscr{X}_{L_1}$ such that $\mathscr{E}_1 \otimes_{L_1} \hat{L} \cong \hat{\mathscr{E}}$ and $F^*\mathscr{E}_1$ is not semi-stable.*

Proof. Let $\hat{\pi} : F^*\hat{\mathscr{E}} \to \hat{\mathscr{L}}$ be a maximal destabilizing quotient of $F^*\hat{\mathscr{E}}$. There exist a models $\mathscr{E}_{k(S)}, \mathscr{L}_{k(S)}$ and $\pi_{k(S)}$ of $\hat{\pi}$ over $\mathscr{X}_{k(S)}$, where $k(S)$ is an extension of finite type of L. The field $k(S)$ is the function field for some algebraic variety S over L.

Shrinking S if necessary, one can assume that $\pi_{k(S)}$ comes from

$$\pi_S : F^*\mathscr{E}_S \to \mathscr{L}_S,$$

where $\mathscr{E}_S$ is a family of stable bundles over $\mathscr{X}_L$ parameterized by S (stability is an open condition). We now choose a closed point $s \in S$ and pull-back the family $\mathscr{E}_S$ under the inclusion $s \hookrightarrow S$. We thus obtain a stable bundle $\mathscr{E}_{L_1}$ over $\mathscr{X}_{L_1}$, where L_1 is the residue field at the point s, which is a finite extension of L. $\square$

Again we take the integral closures A_1 and A'_1 of the discrete valuation rings A and A' in L_1. By the previous lemma we have a stable bundle $\mathscr{E}_1$ and a destabilizing quotient $\mathscr{L}_1$ over $\mathscr{X}_{L_1} = \mathscr{X}'_{L_1}$

$$\pi_{L_1} : F^*\mathscr{E}_1 \to \mathscr{L}_1.$$

After possibly taking a finite extension of L_1, we can assume [La] that $\mathscr{E}_1$ and $\mathscr{L}_1$ have models over $\mathscr{X} \to \mathrm{Spec}(A)$ with $(\mathscr{E}_1)_{\bar{s}}$ semi-stable over $X \times_k \overline{k(s)}$. By semi-continuity, we have

$$\mathrm{Hom}(F^*\mathscr{E}_{1\bar{s}}, \mathscr{L}_{\bar{s}}) \neq 0,$$

which shows that $F^*\mathscr{E}_{1\bar{s}}$ is not semi-stable. $\square$

Remark A.6. The rank 2 bundle $\mathscr{E}_{1\bar{s}}$, constructed in the proof of Theorem 7.4, is actually stable, since a strictly semi-stable rank 2 bundle cannot pull-back to an unstable bundle.

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